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GLOBAL ATTRACTIVITY OF A HIGHER ORDER RATIONAL DIFFERENCE EQUATION

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Abstract

The aim of this paper is to investigate the global attractivity of the recursive sequence $x_{n+1} = \frac{\alpha - \beta x_{n-k}}{\gamma + g(x_n)}$ under specified conditions. We show that the positive (or zero for $\alpha = 0$) equilibrium point of this equation is a global attractor with a basin that depends on certain conditions posed on the coefficients and the function $g(x)$.

1. Introduction

Aboutaleb et al. [1] studied the rational recursive sequence

$$x_{n+1} = \frac{\alpha - \beta x_n}{\gamma + x_{n-1}}, \quad n = 0, 1, \ldots,$$

where the coefficients α, β and γ are non-negative real numbers and

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obtained sufficient conditions for the global attractivity of the positive equilibrium point.

Li and Sun [9] extended the above results to the following rational recursive sequence

$$x_{n+1} = \frac{\alpha - \beta x_n}{\gamma + x_{n-k}}, \quad n = 0, 1, \ldots.$$

El-Owaidy et al. [6] studied the rational recursive sequence

$$x_{n+1} = \frac{\alpha - \beta x_{n-1}}{\gamma + x_n}, \quad n = 0, 1, \ldots,$$

where the coefficients $\alpha, \beta, \gamma > 0$ and obtained sufficient conditions for the global attractivity of the positive equilibrium point with a basin that depends on certain conditions posed on the coefficients. El-Owaidy et al. [5] studied the rational recursive sequence

$$x_{n+1} = \frac{\alpha - \beta x_{n-k}}{\gamma + x_n}, \quad n = 0, 1, \ldots,$$

where the coefficients $\alpha, \beta, \gamma > 0$ and $k = 1, 2, \ldots$ and obtained sufficient conditions for the global attractivity of the positive equilibrium point with a basin that depends on certain conditions posed on the coefficients.

Also, El-Owaidy et al. [4] studied the rational recursive sequences

$$x_{n+1} = \frac{-\alpha x_{n-1}}{\beta \pm x_n}, \quad n = 0, 1, \ldots,$$

where the coefficients $\alpha, \beta > 0$ and obtained sufficient conditions for the global attractivity of the zero equilibrium points with basin that depend on certain conditions posed on the coefficients.

Ahmed [2] studied the rational recursive sequence

$$x_{n+1} = \frac{\alpha - \beta x_{n-1}}{\gamma + g(x_n)}, \quad n = 0, 1, \ldots,$$

where the coefficients α, β and γ are non-negative real numbers and

obtained sufficient conditions for the global attractivity of the zero and positive equilibrium points.

Other related results refer to [3, 7-10].

The aim of this paper is to investigate the global attractivity of the recursive sequence

$$x_{n+1} = \frac{\alpha - \beta x_{n-k}}{\gamma + g(x_n)}, \quad n = 0, 1, \ldots, \tag{1.1}$$

where

$$k = 1, 2, \ldots, \quad \alpha \geq 0, \beta, \gamma > 0 \quad \text{and} \quad \gamma > \beta. \tag{1.2}$$

The function $g(x)$ is a continuous function on $(-\infty, \infty)$ satisfying some conditions which we will explain later.

We show that the positive (or zero for $\alpha = 0$) equilibrium point of Equation (1.1) is a global attractor with a basin that depends on certain conditions posed on the coefficients and the function $g(x)$.

The study of these equations is quite challenging and rewarding and is still in its infancy.

We believe that the nonlinear rational difference equations are of paramount importance in their own right, and furthermore that results about such equations offer prototypes for the development of the basic theory of the global behavior of nonlinear difference equations.

We need the following definitions

Definition 1.1. The equilibrium point $\bar{x}$ of the equation

$$x_{n+1} = F(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n = 0, 1, \ldots \tag{1.3}$$

is the point that satisfies the condition:

$$\bar{x} = F(\bar{x}, \bar{x}, \ldots \bar{x}).$$

Definition 1.2. Let I be an interval of real numbers. The equilibrium point $\bar{x}$ of Equation (1.3) is said to be

(i) **locally stable** if for every $\epsilon > 0$ there exists $\delta > 0$ such that for

all $x_{-k}, \ldots, x_{-1}, x_0 \in I$ with $|x_{-k} - \bar{x}| + \ldots + |x_{-1} - \bar{x}| + |x_0 - \bar{x}| < \delta$, we have $|x_n - \bar{x}| < \epsilon$ for all $n \geq -k$,

(ii) **locally asymptotically stable** if it is locally stable, and if there exists $\gamma > 0$ such that for all $x_{-k}, \ldots, x_{-1}, x_0 \in I$ with $|x_{-k} - \bar{x}| + \ldots + |x_{-1} - \bar{x}| + |x_0 - \bar{x}| < \gamma$, we have $\lim_{n\to\infty} x_n = \bar{x}$.

(iii) **global attractor** if for all $x_{-k}, \ldots, x_{-1}, x_0 \in I$, we have $\lim_{n\to\infty} x_n = \bar{x}$.

(iv) **globally asymptotically stable** if $\bar{x}$ is locally stable and $\bar{x}$ is also global attractor.

(v) **unstable** if $\bar{x}$ is not locally stable.

(vi) **repeller** if there exists $r > 0$ such that if $x_{-k}, \ldots, x_{-1}, x_0 \in I$ and $|x_{-k} - \bar{x}| + \ldots + |x_{-1} - \bar{x}| + |x_0 - \bar{x}| < r$, then there exists $N \geq -k$ such that

$$|x_N - \bar{x}| \geq r.$$

Clearly, a repeller is an unstable equilibrium.

Definition 1.3. Let I be an interval of real numbers and let $f : I^{k+1} \to I$ be a continuously differentiable function. Consider the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n = 0, 1, \ldots \tag{1.4}$$

with $x_{-k}, x_{-k+1}, \ldots, x_0 \in I$. Let $\bar{x}$ be the equilibrium point of equation (1.4). The linearized equation of Equation (1.4) about the equilibrium point $\bar{x}$ is

$$y_{n+1} = c_1 y_n + c_2 y_{n-1} + \ldots + c_{k+1} y_{n-k}, \quad n = 0, 1, \ldots, \tag{1.5}$$

where

$$c_1 = \frac{\partial f}{\partial x_n}(\bar{x}, \bar{x}, \ldots, \bar{x}),\ c_2 = \frac{\partial f}{\partial x_{n-1}}(\bar{x}, \bar{x}, \ldots, \bar{x}), \ldots, c_{k+1} = \frac{\partial f}{\partial x_{n-k}}(\bar{x}, \bar{x}, \ldots, \bar{x}).$$

The characteristic equation of Equation (1.4) is

$$\lambda^{k+1} - c_1\lambda^k - c_2\lambda^{k-1} - \ldots - c_{k+1} = 0. \tag{1.6}$$

We need the following two theorems

Theorem A [7]. (i) *If all roots of Equation* (1.6) *have absolute value less than one, then the equilibrium point* $\bar{x}$ *of Equation* (1.4) *is locally asymptotically stable.*

(ii) *If at least one of the roots of Equation* (1.6) *has absolute value greater than one, then* $\bar{x}$ *is unstable.*

The equilibrium point $\bar{x}$ *of Equation* (1.4) *is called* ***saddle point****, if Equation* (1.6) *has roots both inside and outside the unit disk.*

Theorem B [3]. *Consider the difference equation*

$$y_{n+1} = f(y_n, y_{n-k}), \quad n = 0, 1, \ldots, \tag{1.7}$$

where $k \in \{1, 2, \ldots\}$. *Let* $I = [a, b]$ *be some interval of real numbers and assume that*

$$f : [a, b] \times [a, b] \to [a, b],$$

is a continuous function satisfying the following properties:

(a) $f(u, v)$ *is nonincreasing in each arguments.*

(b) *If* $(m, M) \in [a, b]$ *is a solution of the system*

$$m = f(M, M) \quad \text{and} \quad M = f(m, m),$$

then $m = M$.

Then Equation (1.7) *has a unique positive equilibrium* $\bar{y}$ *and every solution of Equation* (1.7) *converges to* $\bar{y}$.

2. The Case $\alpha > 0$

In this section, we study the behavior of the difference equation

$$x_{n+1} = \frac{\alpha - \beta x_{n-k}}{\gamma + g(x_n)}, \quad n = 0, 1, \ldots, \tag{2.1}$$

where

$$k \in \{1, 2, \ldots\}, \quad \alpha, \beta, \gamma > 0 \quad \text{and} \quad \gamma > \beta. \tag{2.2}$$

The function $g(x)$ is a continuous function on $(-\infty, \infty)$ satisfying

$$\left.\begin{array}{l} \text{(i)} \quad g(x) > 0 \quad \text{for} \quad x > 0. \\ \text{(ii)} \quad g(x) \text{ is increasing on } (-\infty, \infty). \\ \text{(iii)} \ \dfrac{x}{g(x)} \text{ is nondecreasing on } (0, \infty). \end{array}\right\} \tag{2.3}$$

Lemma 2.1. *Assume that* (2.2) *and* (2.3) *hold. Then Equation* (2.1) *has the unique positive equilibrium point.*

Proof. Let $F(x) = x - \dfrac{\alpha - \beta x}{\gamma + g(x)}$. It is clear that $F(x)$ is a continuous function on $[0, \infty)$. Since $F(0) = -\dfrac{\alpha}{\gamma + g(0)} < 0$ and $\lim_{x \to \infty} F(x) = \infty$, then there exists an $\bar{x} \in (0, \infty)$ such that $F(\bar{x}) = 0$. On the other hand, if $x > y$, we can simply show that $F(x) > F(y)$. So, $F(x)$ is increasing on $[0, \infty)$. Hence, $\bar{x}$ is the unique positive equilibrium point of Equation (2.1).

This completes the proof.

Theorem 2.2. *The positive equilibrium point* $\bar{x}$ *of Equation* (2.1) *is a global attractor with a basin*

$$S = [0, \alpha / \beta)^{k+1}.$$

Proof. For $u, v \in [0, \alpha / \beta)$, set

$$f(u, v) = \frac{\alpha - \beta v}{\gamma + g(u)}.$$

Then $f : [0, \alpha / \beta) \times [0, \alpha / \beta) \to [0, \alpha / \beta)$ is a continuous function and is nonincreasing in u and v. Let $(m, M) \in [0, \alpha / \beta)$ is a solution of the system

$$m = f(M, M) \quad \text{and} \quad M = f(m, m),$$

then

$$(\gamma - \beta)(m - M) + mM\left(\frac{g(M)}{M} - \frac{g(m)}{m}\right) = 0.$$

By using the conditions (2.2) and (2.3), we have

$$m = M.$$

By using Theorem B, then $\bar{x}$ of Equation (2.1) is a global attractor with a basin

$$S = [0, \alpha / \beta)^{k+1}.$$

This completes the proof.

Lemma 2.3. *Assume that conditions* (2.2) *and* (2.3) *hold. Let* $\{x_n\}$ *be a solution of Equation* (2.1). *If* $x_{m-k+1}, x_{m-k+2}, \ldots, x_{m+1} \in [0, \alpha / \beta]$ *for some* $m \geq -1$, *then*

$$x_{m+i} \in [C, D] \quad \forall i \geq k + 3,$$

where

$$C = \frac{\alpha - \beta\alpha / \gamma}{\gamma + g(\alpha / \beta)} \quad \textit{and} \quad D = \frac{\alpha}{\gamma}. \tag{2.4}$$

Proof. We can see that $x_{m+2}, x_{m+3}, \ldots, x_{m+k+2} \in [0, \alpha / \gamma]$. Then

$$C = \frac{\alpha - \beta\alpha / \gamma}{\gamma + g(\alpha / \beta)} \leq \frac{\alpha - \beta\alpha / \gamma}{\gamma + g(\alpha / \gamma)} \leq x_{m+k+3} \leq \frac{\alpha}{\gamma} = D.$$

So, the result follows by induction.

Assume that there exists $\tau \geq 2$ such that the following conditions hold

$$\gamma \geq \tau\alpha / \beta \quad \text{and} \quad \alpha \geq \tau\beta^2. \tag{2.5}$$

Lemma 2.4. *Assume that the conditions* (2.5) *hold for some* $\tau \geq 2$. *Let* $\{x_n\}$ *be a solution of Equation* (2.1). *If* $x_{m-k+1}, \ldots, x_m \in [-(\tau - 1)\alpha / \beta, \alpha / \beta]$ *and* $x_{m+1} \in [\max\{-(\tau - 1)\alpha / \beta, g^{-1}(-(\tau - 1)\alpha / \beta)\}, \alpha / \beta]$

for some $m \geq -1$, *then*

$$x_{m+i} \in [C, D] \quad \forall i \geq 2k + 4.$$

Proof. We can see that $x_{m+2}, x_{m+3}, \ldots, x_{m+k+2} \in [0, \alpha / \beta]$. Then the proof follows immediately from Lemma 2.3.

Lemma 2.5. *Assume that the conditions* (2.5) *hold for some* $\tau \geq 2$. *Let* $\{x_n\}$ *be a solution of Equation* (2.1). *Suppose that* $g(-\frac{\alpha}{\beta}) \geq -(\tau - 1)\alpha / \beta$, $x_{m-k+1} \in [-(\tau - 1)\alpha / \beta, \infty)$ *and* $x_{m-k+2}, x_{m-k+3}, \ldots, x_{m+1} \in [\max\{-(\tau - 1)\alpha / \beta, g^{-1}(-(\tau - 1)\alpha / \beta)\}, \alpha / \beta]$ *for some* $m \geq -1$ *such that* $|x_{m-k+1} - g(x_{m+1})| \leq \gamma + \tau\alpha / \beta$, *then*

$$x_{m+i} \in [C, D] \quad \forall i \geq 2k + 5.$$

Proof.

If $\alpha - \beta x_{m-k+1} \geq 0$, then $x_{m+2} \in [0, \alpha / \beta]$.

If $\alpha - \beta x_{m-k+1} < 0$, then $x_{m+2} \in [-\alpha / \beta, 0]$.

In both cases $x_{m+2} \in [-\alpha / \beta, \alpha / \beta]$, which implies that

$$x_{m+3} \in [0, \alpha / \beta].$$

Then by using Lemma 2.4, we have

$$x_{m+i} \in [C, D] \quad \forall i \geq 2k + 5,$$

and then, the proof is complete.

Theorem 2.6. *If there exists* $\tau \geq 2$ *such that conditions* (2.5) *hold, then the positive equilibrium point* $\bar{x}$ *of Equation* (2.1) *is a global attractor with a basin* S *such that if* $g(-\frac{\alpha}{\beta}) \geq -(\tau - 1)\alpha / \beta$, *and* $|x_{-k} - g(x_0)| \leq \gamma + \tau\alpha / \beta$, *then*

$$S = [-(\tau - 1)\alpha / \beta, \infty) \times [\max\{-(\tau - 1)\alpha / \beta, g^{-1}(-(\tau - 1)\alpha / \beta)\}, \alpha / \beta]^k.$$

If else, then

$$S = [-(\tau-1)\alpha/\beta,\ \alpha/\beta)^k \times [\max\{-(\tau-1)\alpha/\beta,\ g^{-1}(-(\tau-1)\alpha/\beta)\},\ \alpha/\beta].$$

Proof. Let $\{x_n\}$ be a solution of Equation (2.1) with initial conditions $(x_{-k}, x_{-k+1}, \ldots, x_0) \in S$. Then by Lemmas 2.4 and 2.5, we have

$$x_n \in [C, D] \quad \forall n \geq 2k+4,$$

where C and D are defined in (2.4).

Set

$$\lambda = \lim_{n\to\infty} \inf x_n \quad \text{and} \quad \Lambda = \lim_{n\to\infty} \sup x_n.$$

Let $\epsilon > 0$ be such that $\epsilon < \min\{(\alpha/\beta) - \Lambda, \lambda\}$. Then there exists $n_0 \in \mathbb{N}$ such that

$$\lambda - \epsilon < x_n < \Lambda + \epsilon \quad \forall n \geq n_0.$$

Hence

$$\frac{\alpha - \beta(\Lambda+\epsilon)}{\gamma + g(\Lambda+\epsilon)} < x_{n+1} < \frac{\alpha - \beta(\lambda-\epsilon)}{\gamma + g(\lambda-\epsilon)} \quad \forall n \geq n_0 + k.$$

Then we get the inequality

$$\frac{\alpha - \beta(\Lambda+\epsilon)}{\gamma + g(\Lambda+\epsilon)} < \lambda < \Lambda < \frac{\alpha - \beta(\lambda-\epsilon)}{\gamma + g(\lambda-\epsilon)}.$$

This inequality yields

$$\frac{\alpha - \beta\Lambda}{\gamma + g(\Lambda)} \leq \lambda \leq \Lambda \leq \frac{\alpha - \beta\lambda}{\gamma + g(\lambda)},$$

which implies that

$$\alpha - \beta\Lambda - \gamma\lambda \leq \lambda g(\Lambda) \leq \Lambda g(\lambda) \leq \alpha - \beta\lambda - \gamma\Lambda.$$

Therefore $(\gamma - \beta)(\Lambda - \lambda) \leq 0$. Hence

$$\lambda = \Lambda = \bar{x}.$$

Then, the proof is complete.

Next, we give some analysis on the semi-cycles of any solution $\{x_n\}$ of Equation (2.1) about $\bar{x}$ with initial conditions $x_{-k}, x_{-k+1}, \ldots, x_0 \in [0, \alpha / \beta]$.

Theorem 2.7. *Assume that the initial conditions* $x_{-k}, x_{-k+1}, \ldots, x_0 \in [0, \alpha / \beta]$ *such that they are not equal to* $\bar{x}$ *together, then the following statements are true*

(i) $\{x_n\}$ *has no* $(k+1)$ *consecutive terms equal to* $\bar{x}$.

(ii) *Every negative semicycle of* $\{x_n\}$ *has at most* $(k+1)$ *terms, while every positive semicycle of* $\{x_n\}$ *has at most* $(2k)$ *terms.*

(iii) $\{x_n\}$ *is strictly oscillatory.*

Proof. (i) If $x_l = x_{l+1} = \ldots = x_{l+k} = \bar{x}$ for some $l \in \mathbb{N}$, then $x_{l-1} = \bar{x}$, which implies that $x_{l-1} = x_{l-2} = \ldots = x_{-k+1} = x_{-k} = \bar{x}$, which is a contradiction.

(ii) Assume that C is a negative semicycle starts with $(k+1)$ terms $x_l, x_{l+1}, \ldots, x_{l+k}$, then $0 \leq x_l, x_{l+1}, \ldots, x_{l+k} < \bar{x}$, which implies that $x_{l+k+1} > \bar{x}$.

Now, suppose that C is a positive semicycle starts with $(k+1)$ terms $x_l, x_{l+1}, \ldots, x_{l+k}$, then $x_l, x_{l+1}, \ldots, x_{l+k} \geq \bar{x}$, which implies that $x_{l+k+1}, x_{l+k+2}, \ldots, x_{l+2k-1} \leq \bar{x}$. Since, all of $x_l, x_{l+1}, \ldots, x_{l+k}$ are not equal to $\bar{x}$ together, then $x_{l+2k} < \bar{x}$.

Note that, if C has $(2k)$ terms, then

$$C = \{x_l = \bar{x}, x_{l+1} = \bar{x}, \ldots, x_{l+k-2} = \bar{x}, x_{l+k-1} > \bar{x}, x_{l+k} = \bar{x}, x_{l+k+1} = \bar{x}, \ldots, x_{l+2k-1} = \bar{x}\},$$

and then $x_{l+2k} < \bar{x}$.

(iii) From (i) and (ii), we get $\{x_n\}$ is strictly oscillatory. This completes the proof.

Remark. When $g(x) = x$, we get the results due to El-Owaidy et al. [5].

3. The Case $\alpha = 0$

In this section, we study the behavior of the difference equation

$$x_{n+1} = \frac{-\beta x_{n-k}}{\gamma + g(x_n)}, \quad n = 0, 1, \ldots, \tag{3.1}$$

where $\beta, \gamma > 0$, $g(x)$ satisfies (2.3) and is differentiable on $(-\infty, \infty)$.

The change of variables

$$x_n = \beta y_n,$$

reduces Equation (3.1) to the difference equation

$$y_{n+1} = \frac{-y_{n-1}}{\frac{\gamma}{\beta} + \frac{1}{\beta} g(\beta y_n)}, \quad n = 0, 1, \ldots. \tag{3.2}$$

Equation (3.2) has two equilibrium points $\bar{y}_1 = 0$ and $\bar{y}_2 = \frac{1}{\beta} g^{-1}(-\beta - \gamma)$.

Theorem 3.1.

(i) *If* $\gamma + g(0) > \beta$, *then* $\bar{y}_1 = 0$ *is locally asymptotically stable.*

(ii) *If* $\gamma + g(0) < \beta$, *then* $\bar{y}_1 = 0$ *is unstable* (*repeller*).

(iii) *If* $\gamma + g(0) = \beta$, *then the linearized stability analysis fails.*

(iv) *The equilibrium point* $\bar{y}_2 = \frac{1}{\beta} g^{-1}(-\beta - \gamma)$ *is unstable* (*saddle point*).

Proof. The linearized equation associated with Equation (3.2) about $\bar{y}_1 = 0$ has the form

$$z_{n+1} + \frac{\beta}{\gamma + g(0)} z_{n-k} = 0, \quad n = 0, 1, \ldots. \tag{3.3}$$

The characteristic equation of Equation (3.3) is

$$\lambda^{k+1} + \frac{\beta}{\gamma + g(0)} = 0.$$

Then the proof of (i), (ii), and (iii) follows immediately from Theorem A.

The linearized equation associated with Equation (3.2) about $\bar{y}_2 = \frac{1}{\beta} g^{-1}(-\beta - \gamma)$ has the form

$$z_{n+1} - \frac{1}{\beta} g^{-1}(-\beta - \gamma) g'(g^{-1}(-\beta - \gamma)) z_n - z_{n-k} = 0. \tag{3.4}$$

The characteristic equation of Equation (3.4) is

$$\lambda^{k+1} - \frac{1}{\beta} g^{-1}(-\beta - \gamma) g'(g^{-1}(-\beta - \gamma)) \lambda^k - 1 = 0. \tag{3.5}$$

Equation (3.5) has $(k+1)$ roots $\lambda_1, \lambda_2, \ldots, \lambda_{k+1}$ such that $|\lambda_1 \lambda_2 \ldots \lambda_{k+1}| = 1$.

From (2.3), $g^{-1}(-\beta - \gamma) \neq 0$, and $g'(g^{-1}(-\beta - \gamma)) \neq 0$. Hence there exists λ_i, λ_j such that $|\lambda_i| > 1$ and $|\lambda_j| < 1$. Then the proof is complete.

In the followings, we assume that $\gamma + g(-\beta) > \beta$.

Lemma 3.2. *Assume that the initial values* $y_{-k}, y_{-k+1}, \ldots, y_0 \in [-1, 0]$.

Then $\{y_{(2k+2)n-k}\}, \{y_{(2k+2)n-k+1}\}, \ldots, \{y_{(2k+2)n}\}$ *is monotonically increasing to zero while* $\{y_{(2k+2)n+1}\}, \{y_{(2k+2)n+2}\}, \ldots, \{y_{(2k+2)n+k+1}\}$ *is monotonically decreasing to zero.*

Proof. Let $y_{-k}, y_{-k+1}, \ldots, y_0 \in [-1, 0]$, then $y_1, y_2, \ldots, y_{k+1} \in [0, 1]$ and $y_{k+2}, y_{k+3}, \ldots, y_{2k+2} \in [-1, 0]$.

By induction, we can see that

$$\{y_{(2k+2)n-k}\}, \{y_{(2k+2)n-k+1}\}, \ldots, \{y_{(2k+2)n}\} \in [-1, 0], \quad n = 1, 2, \ldots$$

and

$$\{y_{(2k+2)n+1}\}, \{y_{(2k+2)n+2}\}, \ldots, \{y_{(2k+2)n+k+1}\} \in [0, 1], \quad n = 1, 2, \ldots.$$

Since

$$\frac{y_{(2k+2)n+1}}{y_{(2k+2)n-2k-1}} = \frac{\beta^2}{(\gamma + g(\beta y_{(2k+2)n}))(\gamma + g(\beta y_{(2k+2)n-k-1}))} < 1,$$

$$y_{(2k+2)n+1} < y_{(2k+2)n-2k-1}, \quad n = 1, 2, \ldots.$$

Similarly, we can see that

$$y_{(2k+2)n+2} < y_{(2k+2)n-2k}, \ldots, y_{(2k+2)n+k+1} < y_{(2k+2)n-k-1}, \quad n = 1, 2, \ldots$$

and

$$y_{(2k+2)n-k} > y_{(2k+2)n-3k-2}, y_{(2k+2)n-k+1} > y_{(2k+2)n-3k-1}, \ldots, y_{(2k+2)n} > y_{(2k+2)n-2k-2}, \quad n = 1, 2, \ldots$$

and the result follows.

Lemma 3.3. *Assume that the initial values* $y_{-k}, y_{-k+1}, \ldots, y_0 \in [0, 1]$.

Then $\{y_{(2k+2)n+1}\}, \{y_{(2k+2)n+2}\}, \ldots, \{y_{(2k+2)n+k+1}\}$ *is monotonically increasing to zero while* $\{y_{(2k+2)n-k}\}, \{y_{(2k+2)n-k+1}\}, \ldots, \{y_{(2k+2)n}\}$ *is monotonically decreasing to zero.*

Proof. The proof is similar to that of Lemma 3.2 and will be omitted.

By using the same technique, we can state the following lemma.

Lemma 3.4. *Assume that the initial values* $y_{-k}, y_{-k+1}, \ldots, y_0 \in [-1, 1]$.

Then $\{y_n\}$ *is monotonically increasing* (*decreasing*) *to zero.*

Theorem 3.5. *The equilibrium point* $\bar{y}_1 = 0$ *of Equation* (2.1) *is a global attractor with a basin*

$$S = [-1, 1]^{k+1}.$$

Proof. The proof follows immediately from Lemmas 3.2, 3.3 and 3.4.

Theorem 3.6. *The equilibrium point* $\bar{y}_1 = 0$ *of Equation* (2.1) *is a global attractor with a basin*

$$S = \{(y_{-k}, y_{-k+1}, \ldots, y_0);\ y_{-k} > 0,\ 0 < y_{-k+1}, \ldots, y_0 < 1 \ \ and \ \ |y_{-k} - \frac{g(\beta y_0)}{\beta}| < \frac{\gamma}{\beta}\}.$$

Proof. Suppose $(y_{-k}, y_{-k+1}, \ldots, y_0) \in S$, then $y_1 \in [-1, 0]$.

Then the proof follows from Theorem 3.5.

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